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FAST TRACK COMMUNICATION

Soliton resonances in a generalized nonlinear Schrödinger equation

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Abstract
It is shown that a generalized nonlinear Schrödinger equation proposed by Malomed and Stenflo admits, for a specific range of parameters, resonant soliton interaction. The equation is transformed to the 'resonant' nonlinear Schrödinger equation, as originally introduced to describe black holes in a Madelung fluid and recently derived in the context of uniaxial wave propagation in a cold collisionless plasma. A Hirota bilinear representation is obtained and soliton solutions are thereby derived. The one-soliton solution interpretation in terms of a black hole in two-dimensional spacetime is given. For the two-soliton solution, resonant interactions of several kinds are found. The addition of a quantum potential term is considered and the reduction is obtained to the resonant NLS equation.

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(Some figures in this article are in colour only in the electronic version)

1. Malomed–Stenflo NLS and RNLS connections

In a search for generalizations of the nonlinear Schrödinger equation which admit Hamiltonian form, Malomed and Stenflo [1] derived the equation

$$\mathrm{i}u_t + u_{xx} + 2p|u|^2u = \left(\bar{c}\frac{u_x^2}{u^2} + c\frac{\bar{u}_x^2}{\bar{u}^2} - 2c\frac{\bar{u}_{xx}}{\bar{u}} - 2c\frac{\bar{u}_x u_x}{\bar{u}u}\right)u \tag{1}$$

with the Hamiltonian density

$$\mathcal{H} = |u_x|^2 - p|u|^4 + c\frac{u}{\bar{u}}\bar{u}_x^2 + \bar{c}\frac{\bar{u}}{u}u_x^2 \tag{2}$$

and the complex parameter $c = c_1 + ic_2$. As was shown by Natterman [2], under the restriction of this parameter to the open disc $|c| < \frac{1}{2}$, equation (1) can be transformed into the NLS equation and, accordingly, is integrable (see also Auberson and Sabatier [3] for real c). Here, it will be shown that (1) is integrable for all values of the complex parameter c, and that, in a specific range of the parameters, it admits resonance solitons.

If we set $u = e^{R+iS}$ then (1) yields

$$-S_t - (1-2c_1)S_x^2 + 2p\,e^{2R} + 2c_2 S_{xx} + (1+2c_1)\left(R_{xx} + R_x^2\right) = 0, \tag{3}$$

$$R_t + (1-2c_1)(S_{xx} + 2R_x S_x) + 2c_2 R_{xx} + 4c_2 R_x^2 = 0 \tag{4}$$

and it is readily seen that the linear transformation

$$S = \hat{S} + \frac{2c_2}{2c_1 - 1}\hat{R}, \qquad R = \hat{R}, \qquad \hat{t} = (2c_1 - 1)t \tag{5}$$

transforms this system into the Madelung form

$$\hat{S}_{\hat{t}} - \hat{S}_x^2 - \frac{2p}{2c_1 - 1}e^{2\hat{R}} - \frac{4|c|^2 - 1}{(2c_1 - 1)^2}\left(\hat{R}_{xx} + \hat{R}_x^2\right) = 0 \tag{6}$$

$$-\hat{R}_{\hat{t}} + (\hat{S}_{xx} + 2\hat{R}_x \hat{S}_x) = 0. \tag{7}$$

Introduction of the new wavefunction

$$\psi = e^{\hat{R} - i\hat{S}} \tag{8}$$

produces the resonant NLS (RNLS) equation of Pashaev and Lee [4],

$$i\psi_{\hat{t}} + \psi_{xx} - \frac{2p}{2c_1 - 1}|\psi|^2\psi = s\frac{|\psi|_{xx}}{|\psi|}\psi, \tag{9}$$

where

$$s = 1 + \frac{4|c|^2 - 1}{(2c_1 - 1)^2}. \tag{10}$$

2. RNLS reductions

2.1. Undercritical case

If $s < 1$ so that $|c| < \frac{1}{2}$, then on rescaling time and the phase of the wavefunction according to

$$\hat{t} = \frac{\tilde{t}}{\sqrt{1-s}}, \qquad \hat{S}(x,t) = \sqrt{1-s}\,\tilde{S}(x,\tilde{t}), \qquad \hat{R}(x,t) = \tilde{R}(x,\tilde{t}), \tag{11}$$

where

$$\sqrt{1-s} = \frac{1 - 4|c|^2}{(1-2c_1)^2}, \tag{12}$$

then we retrieve the usual NLS equation

$$i\tilde{\psi}_{\tilde{t}} + \tilde{\psi}_{xx} + 2p\frac{1-2c_1}{1-4|c|^2}|\tilde{\psi}|^2\tilde{\psi} = 0 \tag{13}$$

in $\tilde{\psi} = e^{\tilde{R} - i\tilde{S}}$. This, as in [2], establishes that when $|c| < \frac{1}{2}$ the Malomed–Steflo equation (1) may be transformed into the standard NLS equation.

2.2. Critical case

If $s = 1$ so that $|c| = \frac{1}{2}$, then on the circle $c_1^2 + c_2^2 = \frac{1}{4}$ equation (9) becomes dispersionless and the resultant NLS equation can be linearized.

2.3. Special case

In the special case when $c_1 = \frac{1}{2}$ and c_2 is an arbitrary real number, the system (3)–(4) reduces [2] to the heat equation

$$-S_t + 2c_2 S_{xx} + 2p\rho + 2\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}} = 0 \tag{14}$$

with density and quantum potential-type sources, together with the heat equation

$$\rho_t + 2c_2\rho_{xx} = 0. \tag{15}$$

for the density $\rho = |u|^2 = \mathrm{e}^{2R}$.

2.4. Overcritical (resonant) case

If $s > 1$, so that $|c| > \frac{1}{2}$ then except on the vertical line $c = \frac{1}{2} + \mathrm{i}c_2$, the RNLS equation cannot be reduced to the NLS form. However, the rescaling

$$\hat{t} = \frac{\tilde{t}}{\sqrt{s-1}}, \qquad \hat{S}(x,t) = \sqrt{s-1}\,\tilde{S}(x,\tilde{t}), \qquad \hat{R}(x,t) = \tilde{R}(x,\tilde{t}), \tag{16}$$

where

$$\sqrt{s-1} = \frac{\sqrt{4|c|^2-1}}{|2c_1-1|} \tag{17}$$

and the introduction of the two real functions E^+, E^- according to

$$E^+ = \mathrm{e}^{\tilde{R}+\tilde{S}}, \qquad E^- = -\mathrm{e}^{\tilde{R}-\tilde{S}} \tag{18}$$

produces the coupled system

$$-E^+_{\tilde{t}} + E^+_{xx} + 2p\frac{2c_1-1}{4|c|^2-1}E^+E^-E^+ = 0, \tag{19}$$

$$E^-_{\tilde{t}} + E^-_{xx} + 2p\frac{2c_1-1}{4|c|^2-1}E^+E^-E^- = 0. \tag{20}$$

2.5. Bilinear representation of the resonant case

The system (19) and (20) can be bilinearized in terms of three real functions G^+, G^- and F where

$$E^+ = \sqrt{\frac{4|c|^2-1}{|p(2c_1-1)|}}\frac{G^+}{F}, \qquad E^- = \sqrt{\frac{4|c|^2-1}{|p(2c_1-1)|}}\frac{G^-}{F} \tag{21}$$

satisfy the system

$$\left(+D_{\tilde{t}} - D_x^2\right)(G^+\cdot F) = 0, \tag{22}$$

$$\left(-D_{\tilde{t}} - D_x^2\right)(G^-\cdot F) = 0, \tag{23}$$

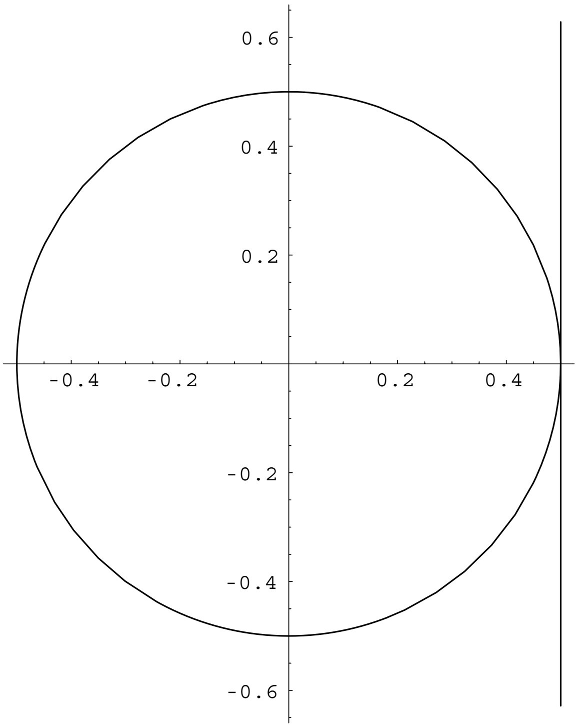

Figure 1. The complex c plane. The region inside the circle $|c| < \frac{1}{2}$ corresponds to the NLS. The circle $|c| = \frac{1}{2}$ is associated with the dispersionless limit of the NLS. Points along the vertical line $x = \frac{1}{2}$ correspond to linear diffusion reductions. The region $|c| > \frac{1}{2}$ corresponds to the resonant case. The right half-plane with $c_1 > \frac{1}{2}$ admits nonsingular solutions for the coupling constant $p < 0$, and for the left half-plane $c_1 < \frac{1}{2}$ for $p > 0$.

$$D_x^2(F \cdot F) = 2\kappa^2 G^+ G^-, \tag{24}$$

where the latter equation shows that

$$-|u|^2 = E^+ E^- = \kappa^2 \frac{4|c|^2 - 1}{|p(2c_1 - 1)|} (\ln F)_{xx}, \tag{25}$$

where $\kappa^2 = \text{sign}\, p((2c_1 - 1)) = \pm 1$.

In the focusing case $p > 0$, for $c_1 > \frac{1}{2}$ we have $\kappa^2 = 1$ while for $c_1 < \frac{1}{2}$ we have $\kappa^2 = -1$ (see figure 1).

In the defocusing case $p < 0$, for $c_1 > \frac{1}{2}$ we have $\kappa^2 = -1$ while for $c_1 < \frac{1}{2}$ we have $\kappa^2 = +1$ (see figure 1).

It is noted that the solution u of the Malomed–Stenflo equation (1) may be written explicitly in a bilinear form as

$$u(x, t) = \left[\frac{4|c|^2 - 1}{|p(2c_1 - 1)|} \frac{1}{F^2} \left(\frac{G^+}{-G^-} \right)^{\mathrm{i}\frac{\sqrt{4|c|^2-1}}{2|2c_1-1|}} \right]^{\frac{2c-1}{2(2c_1-1)}}, \tag{26}$$

where $G^{\pm}(x, \tilde{t}) = G^{\pm}(x, \sqrt{4|c|^2 - 1}t)$, $F(x, \tilde{t}) = F(x, \sqrt{4|c|^2 - 1}t)$, $c = c_1 + \mathrm{i}c_2$.

2.6. Single-soliton solution

For the one-soliton solution we have

$$G^{\pm} = \pm \mathrm{e}^{\eta_1^{\pm}}, \qquad F = 1 - \kappa^2 \mathrm{e}^{\eta_1^+ + \eta_1^- + \phi_{11}}, \qquad \mathrm{e}^{\phi_{11}} = \frac{1}{\left(k_1^+ + k_1^-\right)^2}, \tag{27}$$

where $\eta_1^{\pm} = k_1^{\pm} x \pm \left(k_1^{\pm}\right)^2 \tilde{t} + \eta_1^{\pm(0)}$, and $k_1^{\pm}, \eta_1^{\pm(0)}$ are arbitrary real constants. This solution is regular only if $\kappa^2 < 0$, which corresponds to the cases $p > 0, c_1 < \frac{1}{2}$ or $p < 0, c_1 > \frac{1}{2}$, when $\kappa^2 = -1$ (see figure 1). Here, we focus on this case. From the preceding we have

$$\mathrm{e}^{\hat{R}} = \sqrt{\frac{4|c|^2 - 1}{|p(2c_1 - 1)|}} \frac{\left|k_1^+ + k_1^-\right|}{2\cosh \frac{\eta_1^+ + \eta_1^- + \phi_{11}}{2}}, \qquad \hat{S} = \frac{\sqrt{4|c|^2 - 1}}{|2c_1 - 1|} \frac{\eta_1^+ - \eta_1^-}{2}. \tag{28}$$

Denoting $v \equiv \left(k_1^- - k_1^+\right)\sqrt{4|c|^2 - 1}$, $k \equiv \left(k_1^- + k_1^+\right)/2$ and using $\tilde{t} = \pm\sqrt{4|c|^2 - 1}t$ we obtain a single-soliton solution of the model (1) in the form

$$u(x, t) = \sqrt{\frac{4|c|^2 - 1}{|p(2c_1 - 1)|}} \frac{|k|\,\mathrm{e}^{\mathrm{i}\Phi(x,t)}}{\cosh k(x - vt - x_0)}, \tag{29}$$

where

$$\Phi = \frac{1}{|2c_1 - 1|}\left[-\frac{vx}{2} + \left[(4|c|^2 - 1)k^2 + \frac{v^2}{4}\right]t\right] - \frac{2c_2}{2c_1 - 1}\ln[\cosh k(x - vt - x_0)] + \phi_0. \tag{30}$$

2.7. Hyperbolic metrics and black hole interpretation

Substitution of the Madelung form $u = \mathrm{e}^{R + \mathrm{i}S}$ into the Hamiltonian density (2) yields

$$\mathcal{H} = \left[(1 + 2c_1)R_x^2 + (1 - 2c_1)S_x^2 + 4c_2 R_x S_x\right]\mathrm{e}^{2R} - p\,\mathrm{e}^{4R}. \tag{31}$$

The dispersion is positive definite if $|c| < \frac{1}{2}$ and indefinite when $|c| > \frac{1}{2}$. In the present resonant case, the dispersion is of indefinite sign. Thus in terms of (5)

$$\mathcal{H} = \left[\left(\frac{4|c|^2 - 1}{2c_1 - 1}\right)\hat{R}_x^2 + (1 - 2c_1)\hat{S}_x^2\right]\mathrm{e}^{2\hat{R}} - p\,\mathrm{e}^{4\hat{R}} \tag{32}$$

whence, when $|c| > \frac{1}{2}$ the dispersion is indefinite and it changes sign at points in the spacetime where

$$\hat{R}_x = \pm\frac{1 - 2c_1}{\sqrt{4|c|^2 - 1}}\hat{S}_x. \tag{33}$$

For the one-soliton solution (29) this gives

$$\tanh k(x - vt - x_0) = \pm\frac{v}{2k}, \tag{34}$$

a solution of which exists if $|v| < 2|k|$. As in [4, 5], we can construct a two-dimensional pseudo-Riemannian metric for (19), (20) and the RNLS, namely

$$\mathrm{d}l^2 = \left[(4|c|^2 - 1)\hat{R}_x^2 - (2c_1 - 1)^2\hat{S}_x^2\right]\mathrm{e}^{2\hat{R}}\,\mathrm{d}t^2 - 2\hat{S}_x|2c_1 - 1|\,\mathrm{e}^{2\hat{R}}\,\mathrm{d}x\,\mathrm{d}t - \mathrm{e}^{2\hat{R}}\,\mathrm{d}x^2 \tag{35}$$

so that evolution according to equation (1) implies the two-dimensional spacetime with the constant scalar curvature

$$R = 8p\frac{2c_1 - 1}{4|c|^2 - 1}. \tag{36}$$

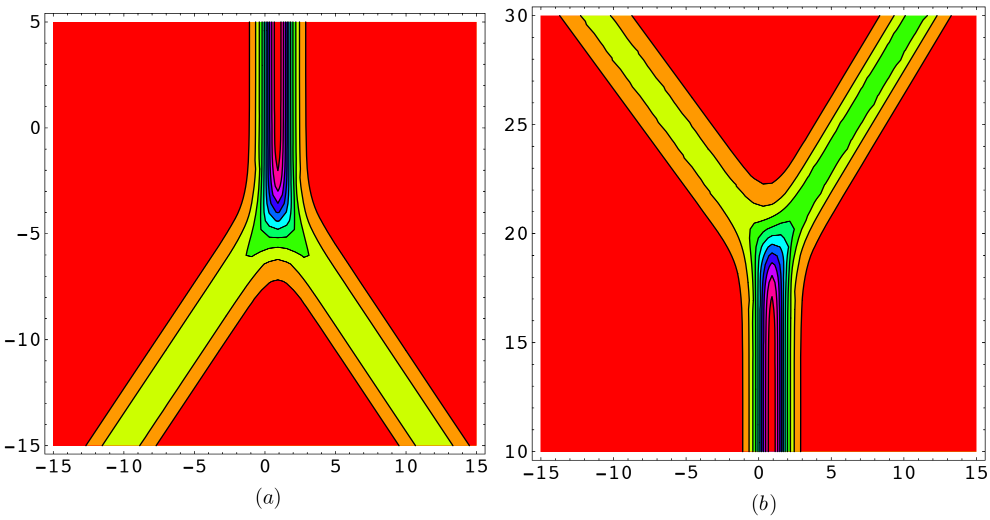

Figure 2. Fusion and fission of two solitons (*a*) fusion of two solitons (*b*) fission of two solitons.

With our choice of parameters, namely $c_1 > \frac{1}{2}$, $p < 0$ or $c_1 < \frac{1}{2}$, $p > 0$, R is negative-valued. The time component of the metric is the dispersion term ϵ_0 for the energy

$$g_{00} = \left[(4|c|^2 - 1)\hat{R}_x^2 - (2c_1 - 1)^2 \hat{S}_x^2\right] \mathrm{e}^{2\hat{R}} = (2c_1 - 1)\epsilon_0. \tag{37}$$

Points where g_{00} vanishes correspond to the event horizon of a black hole. For the one-soliton solution this corresponds to condition (34). Solitons of the equation (1) moving with the velocity $|v| < 2|k|$ correspond to black holes with event horizon dependent on the velocity of the soliton.

2.8. Two-soliton solution

The Hirota bilinear representation (22)–(24) admits two-soliton solutions with

$$G^{\pm} = \pm\left(\mathrm{e}^{\eta_1^{\pm}} + \mathrm{e}^{\eta_2^{\pm}} + \alpha_1^{\pm}\,\mathrm{e}^{\eta_1^{+}+\eta_1^{-}+\eta_2^{\pm}} + \alpha_2^{\pm}\,\mathrm{e}^{\eta_2^{+}+\eta_2^{-}+\eta_1^{\pm}}\right), \tag{38}$$

$$F = 1 + \frac{\mathrm{e}^{\eta_1^{+}+\eta_1^{-}}}{\left(k_{11}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_1^{+}+\eta_2^{-}}}{\left(k_{12}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_2^{+}+\eta_1^{-}}}{\left(k_{21}^{+-}\right)^2} + \frac{\mathrm{e}^{\eta_2^{+}+\eta_2^{-}}}{\left(k_{22}^{+-}\right)^2} + \beta\,\mathrm{e}^{\eta_1^{+}+\eta_1^{-}+\eta_2^{+}+\eta_2^{-}}, \tag{39}$$

where $\eta_i^{\pm} = k_i^{\pm} x \pm \left(k_i^{\pm}\right)^2 \tilde{t} + \eta_i^{\pm(0)}$, $k_{ij}^{ab} = k_i^a + k_j^b$, $(i, j = 1, 2)$, $(a, b = +-)$,

$$\alpha_1^{\pm} = \frac{\left(k_1^{\pm} - k_2^{\pm}\right)^2}{\left(k_{11}^{+-} k_{21}^{\pm\mp}\right)^2}, \qquad \alpha_2^{\pm} = \frac{\left(k_1^{\pm} - k_2^{\pm}\right)^2}{\left(k_{22}^{+-} k_{12}^{\pm\mp}\right)^2}, \qquad \beta = \frac{\left(k_1^{+} - k_2^{+}\right)^2 \left(k_1^{-} - k_2^{-}\right)^2}{\left(k_{11}^{+-} k_{12}^{+-} k_{21}^{+-} k_{22}^{+-}\right)^2}. \tag{40}$$

2.9. Resonance interaction of solitons

In figure 2, fusion and fission of two solitons is shown for the parameter values $k_1^+ = 0.1$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 0$ and large phase shift. The horizontal and vertical axes represent space x and time t coordinates, respectively.

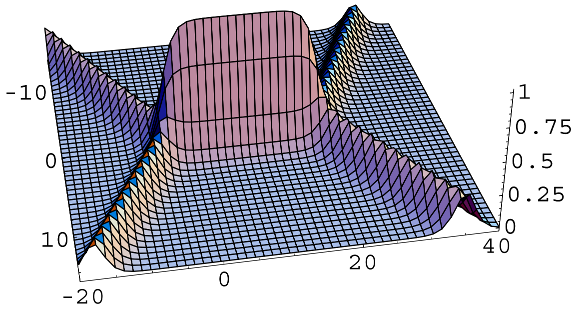

Figure 3. Two-soliton resonant state.

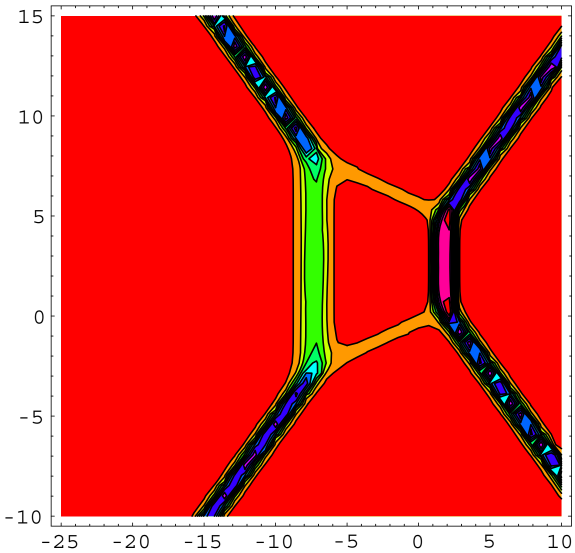

Figure 4. Four-soliton resonance scattering.

In figure 3, the creation of soliton resonance with a finite lifetime is shown. The parameters in this case are the same as above, except for the phase shift $d = 15$.

In figure 4, four virtual soliton resonance scattering is shown for $k_1^+ = 2$, $k_1^- = 1$, $k_2^+ = 1$, $k_2^- = 2$ and $d = 16$.

3. Nontrivial boundary conditions

In the application of the RNLS model to the propagation of solitonic magnetoacoustic waves in [6] the required asymptotic behavior is $|\psi|^2 = \rho \to 1$ at infinity. In this case, we can derive a one-soliton solution of (1) with

$$|u|^2(x,t) = 1 + \frac{v^2 - 4p(1-2c_1)}{4p(1-2c_1)} \operatorname{sech}^2\left[\frac{\sqrt{v^2 - 4p(1-2c_1)}}{2\sqrt{4|c|^2 - 1}}(x + vt + x_0)\right] \tag{41}$$

and the phase

$$S(x,t) = S_0 + 2pt + \frac{c_2}{2c_1 - 1}\ln|u|^2(x,t) \tag{42}$$

$$+\frac{\sqrt{4|c|^2-1}}{2|2c_1-1|}\ln\frac{v+\sqrt{v^2-4p(1-2c_1)}\tanh\left[\frac{\sqrt{v^2-4p(1-2c_1)}}{2\sqrt{4|c|^2-1}}(x+vt+x_0)\right]}{v-\sqrt{v^2-4p(1-2c_1)}\tanh\left[\frac{\sqrt{v^2-4p(1-2c_1)}}{2\sqrt{4|c|^2-1}}(x+vt+x_0)\right]}. \tag{43}$$

It is seen that the velocity of this soliton is bounded below with $|v| > 2|p(1-2c_1)|$. This contrasts with the case of the defocusing NLS equation where the dark soliton velocity is bounded above. Moreover if the soliton of the defocusing NLS is a hole-like (bubble) excitation with $\rho = |u|^2 < 1$, for the Malomed–Stenflo equation this has $\rho = |u|^2 > 1$. It is noted that the two-soliton solution can be constructed alternatively via a Backlund–Darboux transformation [6]. Solutions of the RNLS equation with nontrivial boundary conditions have been investigated by Lee and Pashaev in [7]. These results may be carried over '*mutatis mutandis*' to the Malomed–Stenflo equation (1).

4. Conclusion

It has been established that the generalized nonlinear Schrödinger equation (1) introduced in [1], for a specific range of parameters, admits resonant soliton interaction. Indeed, a natural integrable extension of this equation is suggested, namely

$$\mathrm{i}u_t + u_{xx} + 2p|u|^2u = \left(\bar{c}\frac{u_x^2}{u^2} + c\frac{\bar{u}_x^2}{\bar{u}^2} - 2c\frac{\bar{u}_{xx}}{\bar{u}} - 2c\frac{\bar{u}_x u_x}{\bar{u}u}\right)u + 4\nu\frac{|u|_{xx}}{|u|}u \tag{44}$$

corresponding to the addition of a 'quantum potential' term with strength ν. This extension can be motivated in an information theory context to reflect uncertainty conditions in the measurement process and described by the Fisher measure [8]. The generalized NLS equation (44) is Hamiltonian with

$$\mathcal{H} = |u_x|^2 - p|u|^4 + c\frac{u}{\bar{u}}\bar{u}_x^2 + \bar{c}\frac{\bar{u}}{u}u_x^2 - 4\nu(|u|_x)^2. \tag{45}$$

Following the same procedure as that for (1), reduction may be made to the RNLS form (9) but now with the parameter

$$s = 1 + \frac{4|c|^2 - 1 - 4\nu(2c_1 - 1)}{(2c_1 - 1)^2}. \tag{46}$$

The reductions of the extended model equation (44) then depend on both the complex parameter $c = c_1 + \mathrm{i}c_2$ and the real quantum potential strength ν. In geometrical terms, the circle $|c| = \frac{1}{2}$ in figure 1 is modified by the presence of the additional parameter ν to become

$$(c_1 - \nu)^2 + c_2^2 = \left(\nu - \tfrac{1}{2}\right)^2. \tag{47}$$

The region inside this circle corresponds to the NLS reduction, while the outside corresponds to the resonant NLS case. It is noted that when $\nu = \frac{1}{2}$, the disc shrinks to a point and no reduction to the classical NLS is possible. In this case

$$s = 1 + \frac{(2c_1 - 1)^2 + 4c_2^2}{(2c_1 - 1)^2} \tag{48}$$

whence $s > 1$ and the model equation (44) is necessarily of resonant type.

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